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given in the following table which contains in its first column a number F_n, in its second column a prime factor of F_n and in its third column a binomial number having the same factor, the table being complete as to all known prime factors of composite F_n:

F_5	$2^7 \cdot 5 + 1$	$40^8 + 1$
F_5	$2^7 \cdot 52347 + 1$	$418776^8 + 1$
F_6	$2^8 \cdot 3^2 \cdot 7 \cdot 17 + 1$	$1071^8 + 1$
F_6	$2^8 \cdot 5 \cdot 52562829149 + 1$	$262814145745^8 + 1$
F_9	$2^{16} \cdot 37 + 1$	$37^{32} + 1$
F_{11}	$2^{13} \cdot 3 \cdot 13 + 1$	$1248^{256} + 1$
F_{11}	$2^{13} \cdot 7 \cdot 17 + 1$	$3808^{256} + 1$
F_{12}	$2^{14} \cdot 7 + 1$	$448^{512} + 1$
F_{12}	$2^{16} \cdot 397 + 1$	$397^{256} + 1$
F_{12}	$2^{16} \cdot 7 \cdot 139 + 1$	$973^{256} + 1$
F_{18}	$2^{20} \cdot 13 + 1$	$208^{2^{14}} + 1$
F_{23}	$2^{25} \cdot 5 + 1$	$2560^{2^{19}} + 1$
F_{36}	$2^{39} \cdot 5 + 1$	$640^{2^{31}} + 1$
F_{38}	$2^{41} \cdot 3 + 1$	$1536^{2^{33}} + 1$
F_{73}	$2^{75} \cdot 5 + 1$	$10240^{2^{67}} + 1$

In addition to the information furnished by these fifteen Fermat prime factors it is known that F_0, F_1, F_2, F_3, F_4 are prime and that F_7 and F_8 are composite, as we have said above.

17. In a similar way (3), (4) and (10) yield factors of other simple binomial expressions. We shall merely illustrate this by means of two examples involving (3). Taking the first factor $2^7 \cdot 5 + 1$ of F_5 we have $2 \cdot 32 - 9 \cdot 7 = 1$. Hence $s = 9$, $r = 2$. Therefore

$$5^9 + 2 \equiv 0 \bmod 2^7 \cdot 5 + 1.$$

Again, taking $n = 9$ we may write the given factor in the form $2^{15} \cdot 74 + 1$. Then we have $8 \cdot 512 - 273 \cdot 15 = 1$; whence

$$74^{273} + 2 \equiv 0 \bmod 2^{16} \cdot 37 + 1.$$

SOME GEOMETRICAL RELATIONS OF THE PLANE, SPHERE, AND TETRAHEDRON.

By PHILIP FRANKLIN, College of the City of New York.

Professor Chrystal has remarked that mathematical works should be read backwards as well as forwards, *i. e.*, advanced notions of mathematics often throw light on elementary problems. While this "back tracking" is frequently brought out in analysis, it is seldom emphasized in connection with plane and solid geometry.

Many theorems of plane geometry have solid analogues which are but seldom developed. Especially is this true in the case of constructions which are slighted in solid geometry, owing to their theoretic nature. (The only way of actually drawing them being by the very indirect methods of descriptive geometry.)

For example, the problem of the "tangencies"—the construction of a *circle* in a plane from *three* elements, these being *points,* tangent *lines* or tangent *circles* —with its ten cases easily leads to the determination of a *sphere* from *four* elements, these being *points,* tangent *planes* or tangent *spheres.* There are fifteen cases in all, which are reduced to six by the following two principles, analogous to those used for the plane problem.

PRINCIPLE I.—If the given elements are only spheres and planes, and if the radii of the spheres be changed by that of one of the spheres, and if the planes be translated normal to themselves by a like amount, obviously the sphere determined by the new elements will be concentric with that determined by the original set. This principle serves to replace one of the tangent spheres by a point.

PRINCIPLE II.—If a sphere is tangent to two other spheres (sphere and plane) and goes through a given point, it goes through the inverse of this point, the center of inversion is a center of similitude of the two spheres (sphere and plane) and the radius of inversion is such as to make one of the spheres the inverse of the other (or of the plane). This also replaces one of the given spheres by a point.

The constructions follow; the proofs will be easily supplied by the reader familiar with the plane problem.

1. *Point, Point, Point, Point.*—The center of the required sphere is the intersection of the perpendicular bisecting planes of the joins of the points two and two. (Circum-sphere of a tetrahedron.) One solution.

2. *Plane, Plane, Plane, Plane.*—The center of the required sphere is the intersection of the planes bisecting the dihedral angles formed by the given planes. (In-sphere of a tetrahedron.) One solution.

3. *Point, Point, Point, Plane.*—Construct the perpendicular bisecting planes of the joins of the points intersecting in a line. Draw a sphere with center on this line to touch the given plane, and with the intersection of this line and the given plane as a "center" multiply this sphere to pass through the given points. Two solutions.

4. *Plane, Plane, Plane, Point.*—Draw the bisecting planes of the dihedral angles of the given planes, and draw a sphere with center any point on their line of intersection to touch the planes, then with the point of intersection of the given planes as a "center," multiply this sphere to pass through the given point. Two solutions.

5. *Point, Point, Point, Sphere.*—Draw any sphere through the three points to intersect the given sphere in a circle, and from the line of intersection of the plane of this circle with the plane of the three points draw a tangent plane to the given sphere, locating the point of contact of the required sphere. Then use (1). Two solutions.

6. *Plane, Plane, Plane, Sphere.*—Reduce by Principle I to (4). Four solutions.

7. *Plane, Plane, Point, Point.*—Locate the locus of the center as the intersection of the bisecting plane of the given dihedral angle and the perpendicular bisector of the join of the two points, draw a sphere with center on this line to touch the given planes, and with the intersection of the line of centers with the given planes as "center" multiply it to pass through the given points. Two solutions.

8. *Point, Point, Plane, Sphere.*—Reduce by Principle II to (3). Four solutions.

9. *Plane, Plane, Point, Sphere.*—Reduce by Principle II to (7). Four solutions.

10. *Point, Point, Sphere, Sphere.*—Reduce by Principle II to (5). Four solutions.

11. *Plane, Plane, Sphere, Sphere.*—Reduce by Principle I to (9). Eight solutions.

12. *Plane, Point, Sphere, Sphere.*—Reduce by Principle II to (3). Eight solutions.

13. *Point, Sphere, Sphere, Sphere.*—Reduce by Principle II to (5). Eight solutions.

14. *Plane, Sphere, Sphere, Sphere.*—Reduce by Principle I to (12). Eight solutions.

15. *Sphere, Sphere, Sphere, Sphere.*—Reduce by Principle I to (13). Sixteen solutions.

This last case can also be solved by a direct construction after Gergonne's construction for a circle tangent to three circles,[1] and, like it, in degenerate forms includes many remaining cases of the problem.

With center the intersection of the four radical planes of the spheres, and a radius equal to the tangent from this point to one of the given spheres, describe the sphere orthogonal to the given four spheres. Draw the planes of intersection of this sphere and each of the given spheres intersecting an "axial plane of similitude" of the spheres in four lines. Tangent planes from these lines to the corresponding spheres determine eight points of contact. The two spheres through these four and four are tangent to the given spheres. The eight "axial planes of similitude" determine the sixteen solutions.

The properties of plane triangles are often generalized for spherical triangles. These results, however, are unlike the ordinary ones of geometry, since they often involve trigonometric functions, replace straight lines by great circles, and are really a representation in Euclidean three-dimensional space of a non-euclidean plane.

The tetrahedron is the more natural spacial analogue of the plane triangle, but its properties are seldom brought out. The necessary and sufficient conditions for the co-planarity of six points on the different edges or for the concurrency of six planes through the different edges is easily obtained by applying Ceva's or Menelaus' theorem to three different faces, the condition being three equations in each case. The generalized "Ceva's theorem" might be used to prove the existence of the "axial planes of similitude" of four spheres mentioned above.

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[1] "Recherche du cercle qui en touche trois autres dans un plan," *Annales de Mathématiques* vol. 7, 1817.—Editor.

The existence of a median point (center of mean position or center of mass of the four vertices), *i. e.*, the intersection of the lines joining the centroids of the faces with the opposite vertices, and of the in- and circum-centers is usually proved, but the analogy is carried no further.

If an extension of an ortho-center be attempted, whether we define an altitude as a perpendicular from a vertex to the opposite face or as the join of a vertex with the ortho-center of the opposite face, we find that in general such altitudes do not intersect. The necessary and sufficient condition for either of these sets of "altitudes" being concurrent is the coincidence of the foot of one perpendicular with the ortho-center of the face to which it is perpendicular, in which case both sets of altitudes coincide, and the tetrahedron has its opposite edges perpendicular. Such a tetrahedron has many interesting properties.

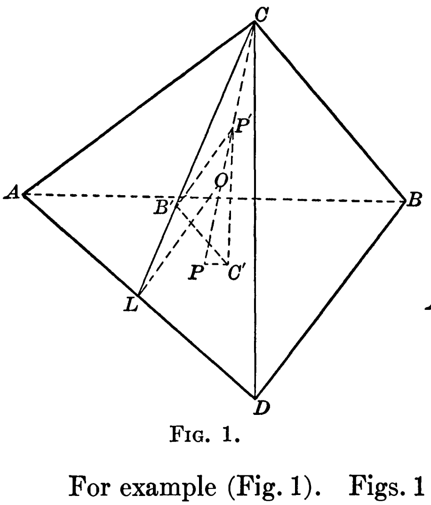

FIG. 1.

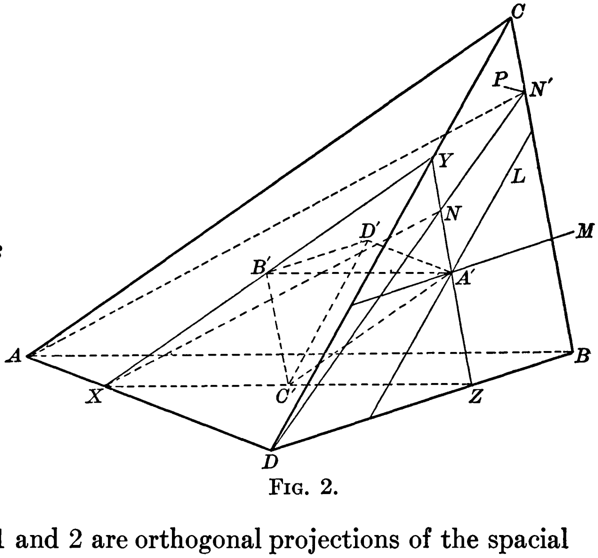

FIG. 2.

For example (Fig. 1). Figs. 1 and 2 are orthogonal projections of the spacial configurations they represent), if $ABCD$ is such a tetrahedron, the four centroids of the faces (as B', C'), the four feet of the altitudes (as P), and the four points dividing the segments of altitudes beyond the ortho-center in the ratio of 1 : 2 (as P', such that $CP' = 2P'O$) all lie on a sphere, which might be called the "twelve-point" sphere after the "nine-point" circle.

Proof. $AO \perp$ Plane DBC

$DO \perp$ Plane ABC

$\therefore$ Plane $AOD \perp$ Planes DBC and ABC or to CB their intersection

$\therefore LO \perp CB$

But $CB' : B'L = CP' : P'O = BC' : C'L = 2 : 1$.

$\therefore LO \parallel B'P'$ and $CB \parallel B'C'$.

$\therefore B'P' \perp B'C'$.

But $PP' \perp PC'$;

$\therefore$ Sphere on diameter $C'P$ passes through B' (and similarly through remaining centroids of faces) and also through P and P'. By a similar proof, the sphere through the centroids may be shown to pass through the other points corresponding to P and P'.

Such a tetrahedron also possesses an "Euler" line. Since $A'B'C'D'$ is similarly situated with regard to $ABCD$ with respect to the median point M, T the center of the twelve-point sphere, K the circum-center and M are collinear, and $KM : MT = 3 : 1$. Also since $OP' : OC = 1 : 3$, etc., the twelve-point sphere and the circum-sphere are similarly situated with regard to 0, and therefore O, T and K are collinear and $OT : OK = 1 : 3$. Also from the similar situation of $A'B'C'D'$ and $ABCD$ O, M and Q the intersection of the perpendiculars at the centroids of the faces are collinear and $MQ : MO = 1 : 3$. Therefore

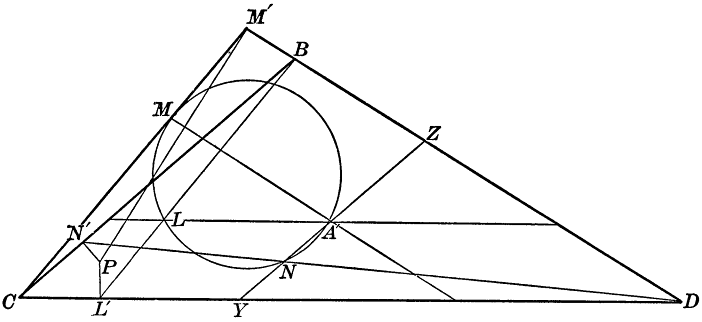

FIG. 3.

O = the ortho-center,
T = the center of the twelve-point sphere,
M = the median point,
Q = the intersection of the perpendiculars at the centroids of the faces and
K = the center of the circum-sphere

are on line in the above order, and the segments $OT : TM : MQ : QK = 2 : 1 : 1 : 2$.

Conversely, solid constructions may give rise to plane theorems. If the centroids of the faces of any tetrahedron $ABCD$ (Fig. 2) $A'B'C'D'$ be located and a sphere passed through them, this sphere cuts plane $A'B'C'$ in the nine-point circle of triangle XYZ and therefore passes through N the foot of the altitude of this triangle on YZ. Similar points may be located corresponding to faces $A'B'D'$ and $A'C'D'$. But DN cuts BC in N', the foot of the A-altitude of triangle ABC, and N' is also the foot of a perpendicular on BC from P, the foot of a perpendicular from A to BDC.

Since for different tetrahedra P becomes any point in plane BCD, the following theorem is obtained (Fig. 3): If from a point (P) perpendiculars be drawn to the sides of a triangle (BCD) produced if necessary, and their feet ($L'M'N'$) be joined to the opposite vertices, the three intersections of these lines with the parallels to the sides through the median point (LMN) and the median point (A') lie on a circle whose center divides the join of the circum-center of the original triangle and the center of mean position of the three vertices of the triangle and the arbitrary point externally in the ratio 1 : 4. The second part of the theorem follows from the fact that $B'C'D'$ and BCD (Fig. 2) are similarly situated with respect to the center of mass of the four vertices of the tetrahedron, and the orthogonal projection of the circum-center of $A'B'C'D'$ on plane BCD coincides with that of the circum-center of triangle $B'C'D'$.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

NEW QUESTIONS.

37. Criticise the following as fundamental definitions of elementary geometry:

A plane surface is the limit approached by a finite portion of the surface of a sphere as the radius increases without limit.

A straight line is the limit approached by a finite portion of the circumference of a circle as the radius increases without limit.

38. As the several courses in secondary and collegiate mathematics are now taught there is a noticeable difference of treatment with respect to the relative emphasis placed on logical accuracy and on development of technique. In elementary algebra, technique predominates, while in plane and solid geometry and advanced algebra logic is more strongly stressed. Trigonometry, analytic geometry, and the calculus are less easily classified; but there is at least a general tendency to emphasize logic in analytic geometry and technique in the calculus. It is suggested that a general appraisal of the reasons for this difference in treatment, its value, and possible alterations, would be helpful.

DISCUSSIONS.

THE EXISTENCE OF CUSPS ON THE EVOLUTE AT POINTS OF MAXIMUM AND MINIMUM CURVATURE ON THE BASE CURVE.

By G. H. LIGHT, University of Colorado.

The purpose of this paper is to show that whenever a curve $F(x, y) = 0$ has a point of maximum or minimum curvature the evolute has at the correspond-